Dynamic Stability of Cylindrical Shells under Static and Periodic Axial and Radial Loads

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Several cases of the dynamic stability problem for cylinders are solved explicitly. Various combinations of the static and periodic radial and axial loads are considered. The regions of instability and stability are established directly in terms of the geometry of the cylinder, load intensity, and frequency. All the numerical results obtained here tend to indicate that a cylinder designed according to the static buckling load can withstand an additional periodic load. It was found that the ratio of additional periodic load to the static buckling load was greater for a thin cylinder than for a thicker cylinder of the same radius. It also was found that the resistance of a cylinder to loads in excess of the static buckling load increases with increasing frequency of loading.

Nomenclature

A, B	=	see Eq. (11)
a, b, e, f	=	see Eq. (7)
a_i, b_i	=	see Eq. (14)
C	=	viscous damping coefficient
D	=	$Eh^3/12(1 - \nu^2)$
E	=	Young's modulus
g	=	see Eq. (4)
h	=	shell thickness
i	=	number of half waves in the axial direction
k	=	number of waves in the circumferential direction
l	=	length of the cylinder
n	=	$i\pi r/l$
N_1, N_2	=	finite axial and circumferential compressive membrane forces
N_0, P_0	=	applied axial load and radial pressure, Eq. (7)
N_{cr}, P_{cr}	=	classical buckling axial load and radial pressure of a long cylinder, Eq. (11)
P_s	=	static buckling pressure for a finite cylinder with simply supported edges
r	=	mean radius of the cylindrical shell
t	=	$\omega\tau/2$
u, v, w	=	displacement component
x, y, z	=	coordinates
α	=	x/r
β	=	$y/4$
δ	=	$c/2\rho$
η	=	see Eq. (5)
μ	=	see Eq. (12)
ξ	=	see Eq. (2)
ν	=	Poisson's ratio = 0.3
ρ	=	mass of shell wall per unit area of middle surface
τ	=	time
ϕ	=	see Eq. (12)
ω	=	angular velocity of the periodic force
ω_0	=	$[Eh/\rho r^2(1 - \nu^2)]^{1/2}$
∇^2	=	$(\partial^2/\partial\alpha^2) + (\partial^2/\partial\beta^2)$

I. Introduction

A CYLINDRICAL shell subjected to periodic loads can have various periodic oscillations in addition to those oscillations having the same period as the external force. Furthermore, the same type of oscillation may exist for the system with different initial conditions.[2] If the oscillation is the "harmonic" or the "subharmonic" type, it will be sustained by the system, provided that damping does not exist. For other types, the amplitude of oscillation may diminish with time (stable) or increase with time (unstable).

Many factors contribute to the characteristic of the oscillation: the geometry of the shell, the load parameters, and the modes of oscillation. If the periodic load can be represented by several harmonic terms, the equation of motion is known as Hill's equation. If the load expression has only one harmonic term, the equation is reduced to the well-known equation introduced by Mathieu[3] in 1868 in his study of the vibrational modes of an elliptical membrane:

$$(d^2\eta/dt^2) + (A + B\cos 2t)\eta = 0$$

The stability characteristic of the solution of Mathieu's equation is known only in terms of those coefficients of the dependent variables A and B. In 1925, Ince[4] made a stability chart showing the regions of parameters A and B of Mathieu's equation which yield stable or unstable solutions. In 1949, Markov[5] treated the problem of dynamic stability of anisotropic cylindrical shells subjected to axial and radial varying loads. In 1950, Oniashvili[6] studied a similar problem for shallow cylindrical shells, and in 1956, Bolotin[7] published a book on the dynamic stability of elastic systems. Previously, studies had been confined to the formulation of an equation of motion like Mathieu's equation, giving the impression that, since it had been treated by Ince and others before 1925, the problem of stability had been solved.

However, a second look reveals that, for a given set of A and B taken from Ince's stability chart, Mathieu's equation really does not indicate whether or not a cylinder can be stable. A set of A and B associated with the system is not unique (A and B contain parameters of many possible variations), and the discussion of its stability in terms of A and B alone does not bear any physical significance (see Sec. II). To solve the stability problem, all the parameters should be considered independently and the final results expressed directly in terms of all the parameters instead of only A and B. This involves appreciable numerical work, but only by carrying out a detailed study will it be possible to extend the theory for practical use.

In this report, the problem of the dynamic stability of a cylinder is determined directly in terms of its geometry, load intensity, and frequency. The load is assumed to consist of one static and one periodic load component only. (If the loading consists of a Fourier expansion of many harmonic terms, Mathieu's equation is replaced by a Hill's-type equa-

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tion of motion which requires special treatment. Some useful information concerning it, however, may be found in Ref. 1. This case will not be considered here.) It also is assumed that the frequency of the periodic load is appreciably lower than the frequency of stress waves in the cylinder.

The problems under consideration include 1) closed cylinders under static and periodic external pressure, 2) cylinders under static and periodic radial pressure, 3) cylinders under static and periodic axial loads, and 4) interaction curve for cylinders under varying axial and radial loads. The numerical results obtained are presented as curves in Figs. 1–11.

II. Equations of Motion

The displacement of a cylinder under external loads may be resolved into two components. One component is of cylindrical form. For example, a long cylinder under oscillatory uniform radial load will contract or expand radially according to the sense of the load but always will maintain its shape as a circular cylinder. The other component is called the variational displacement, differing from the cylindrical form, e.g., having a shape that is sinusoidal both in the axial and the circumferential directions.

The approximate equilibrium equations of motion of shell element in terms of these two components of displacement can be separated (see Ref. 7, Chap. 22) so that the governing equation contains the variational displacement only. Let the axial load be designated by $N_1(\tau)$ and the radial pressure by $P(\tau)$, using nondimensional coordinates $\alpha = x/r$ and $\beta = y/r$ (Fig. 12); the three equations of equilibrium of a shell element in terms of the variational displacement u, v, and w may be recuced to a single eighth-order differential equation in[8, 9]

$$(\nabla^2 + 1)^2\nabla^4 w - 2(1-\nu)\frac{\partial^2}{\partial\alpha^2}\left(\frac{\partial^2}{\partial\alpha^2} - \frac{\partial^2}{\partial\beta^2}\right)\nabla^2 w + 12(1-\nu^2)\left(\frac{r}{h}\right)^2\frac{\partial^4 w}{\partial\alpha^4} + \frac{r^4}{D}\left(\rho\frac{\partial^2}{\partial\tau^2} + C\frac{\partial}{\partial\tau}\right)\nabla^4 w + \frac{r^2}{D}\left[N_1\frac{\partial^2}{\partial\alpha^2} + N_2\left(\frac{\partial^2}{\partial\beta^2} + 1\right)\right]\nabla^4 w = 0 \quad (1)$$

where $N_2 = P(\tau)r$, ρ is the mass of shell wall per unit area of middle surface, C is the viscous damping coefficient, and

$$\nabla^2 = \frac{\partial^2}{\partial\alpha^2} + \frac{\partial^2}{\partial\beta^2} \qquad \nabla^4 = \nabla^2\nabla^2$$

In Eq. (1), the inertia force due to w is considered, but those forces due to the axial and circumferential displacements are ignored because they are of a higher order.

Assuming that both the ends of the cylinder are simply supported, the mode of the variational radial displacement can be taken as

$$w = \xi(\tau)\sin n\alpha \cos k\beta \quad (2)$$

Substituting this expression into Eq. (1) leads to the following second-order linear differential equation for ξ:

$$\frac{d^2\xi}{d\tau^2} + 2\delta\frac{d\xi}{d\tau} + \frac{Dg}{\rho r^4}\xi - \frac{1}{\rho r^2}[n^2 N_1(\tau) + (k^2-1)N_2(\tau)]\xi = 0 \quad (3)$$

Here, $2\delta = c/\rho$, and

$$g = \frac{(n^2+k^2-1)^2(n^2+k^2)^2 + 2(1-\nu)n^2(n^4-k^4) + 12(1-\nu^2)(r/h)^2 n^4}{(n^2+k^2)^2} \quad (4)$$

Finally, by introducing a new variable η such that

$$\xi = e^{-\delta\tau}\cdot\eta \quad (5)$$

Eq. (3) is transformed to the standard form

$$\frac{d^2\eta}{d\tau^2} + \left\{\frac{Dg}{\rho r^4} - \frac{1}{\rho r^2}[n^2 N_1(\tau) + (k^2-1)N_2(\tau)] - \delta^2\right\}\times \eta = 0 \quad (6)$$

If the inertia term and the δ term are dropped, and N_1 and N_2 are independent of time, then Eq. (6) reduces to the equation for static buckling problems.

The general expression for the loading is assumed to be

$$N_1 = N_0(a\cos\omega\tau + e)$$
$$N_2 = rP = rP_0(b\cos\omega\tau + f) \quad (7)$$

where N_0, P_0, a, b, e, and f are constant coefficients. Some of the coefficients may be taken as equal to zero in special cases. In general, the frequencies for the periodical components of N_1 and N_2 may differ. Here, consideration will be given only to the case of identical frequencies, which in practice can be the case of cylinders under hydraulic pressure. Substituting the loading expression, Eq. (7), into Eq. (6) yields

$$\frac{d^2\eta}{d\tau^2} + \left\{\frac{Dg}{\rho r^4} - \delta^2 - \frac{1}{\rho r}\left[\frac{n^2}{r}eN_0 + (k^2-1)fP_0\right] - \frac{1}{\rho r^2}[n^2 a N_0 + r(k^2-1)bp_0]\cos\omega\tau\right\}\eta = 0 \quad (8)$$

For convenience, τ is taken as the transformation

$$\tau = 2(t/\omega) \quad (9)$$

and Eq. (8) becomes

$$(d^2\eta/dt^2) + (A + B\cos 2t)\eta = 0 \quad (10)$$

where

$$A = -4\left(\frac{\delta}{\omega}\right)^2 + \frac{1}{3}\left(\frac{h}{r}\right)^2\left(\frac{\omega_0}{\omega}\right)^2\times\left\{g - 4[3(1-\nu)^2]^{1/2}n^2\frac{r}{h}e\frac{N_0}{N_{cr}} - 3(k^2-1)f\frac{P_0}{P_{cr}}\right\}$$

$$B = -4\left(\frac{\omega_0}{\omega}\right)^2\left[\frac{(1-\nu^2)^{1/2}}{3}n^2\left(\frac{h}{r}\right)a\frac{N_0}{N_{cr}} + \frac{k^2-1}{4}\left(\frac{h}{r}\right)^2 b\frac{P_0}{P_{cr}}\right] \quad (11)$$

$$\omega_0^2 = \frac{Eh}{(1-\nu^2)\rho r^2}$$

and N_{cr} is the classical axial buckling force and P_{cr} the classical buckling pressure of a long cylinder, that is,

$$N_{cr} = \frac{Eh^2}{r[3(1-\nu^2)]^{1/2}} \qquad P_{cr} = \frac{Eh^3}{4(1-\nu^2)r^2}$$

Equation (10) is Mathieu's equation;[3] the solution may be bounded or unbounded, depending on the parameters A and B. The stability characteristic of the solution will be discussed briefly in the next section. However, it is noteworthy to state here that A and B contain 12 parameters, all of which are fixed by geometry and loading except for the wave parameters n and k. In seeking a possible mode of instability, a large number of values of n and k must be examined. The maximum stable load for a cylinder is determined by the most critically selected set of n and k besides the other given parameters.

III. Stability of Mathieu Equation[10]

By Floquet's theory,[11] the general solution of Eq. (10) is given by

$$\eta = c_1 e^{\mu t}\,\phi(t) + c_2 e^{-\mu t}\,\phi(-t) \tag{12}$$

where μ is called the characteristic exponent, and $\phi(t)$ is a periodic function of t with period π or 2π. In numerical work it is possible to arrange that μ be either real or pure imaginary, but not otherwise complex.

The stability of solutions of η is determined by the following criteria:

1) A solution is said to be unstable if it tends toward $\pm\infty$ as $t \to +\infty$ (μ = any real number).†

2) A solution is stable if it tends to zero or remains bounded as $t \to \infty$ ($\mu = i\theta$).

3) A solution with period π or 2π is said to be neutral, a special case of the second criterion ($\mu = i\theta$, $\theta = 0$ or integer). The solutions in case 3 are called Mathieu functions. If B is given, A is a definite quantity for each function and is called the characteristic number of that function. The stability chart by Ince provides the A-B curves for Mathieu functions up to the sixth order. A similar chart is presented in Figs. 1 and 2. In Fig. 1, the shadowed areas between curves are unstable regions, i.e., points A, B in those regions yield unstable solutions for the Mathieu equation. Otherwise, the regions outside the shadowed areas are stable.

For a given pair of A and B, the characteristic exponent μ is a definite number. In the unstable region, if all the points A, B having the same value of μ are connected, a curve is obtained called the iso-μ curve (shown by one of the broken curves in Fig. 1). The iso-μ curves for the first, second, and third unstable regions may be found from Refs. 10 and 1. New iso-μ curves are added to the third, fourth, fifth, and sixth unstable regions for $\mu = 0.05$ (Fig. 2) by using the method of the infinite continued fraction (Ref. 10, p. 107). The use of the iso-μ curves will be clear immediately when it comes to the solution of the problem which takes into consideration the effect of viscous damping.

IV. Stability Characteristics of the Equation of Motion

After a brief discussion of the stability of the Mathieu equation, the problem of immediate interest is the stability characteristics of Eq. (3) [later reduced to the Mathieu Eq. (10) by using Eq. (5)]. It is clear from Eqs. (5, 9, and 12) that

$$\xi = e^{-\delta\tau}\, e^{\mu t}\,\phi(t) = e^{(-\delta+\mu\omega/2)\tau}\,\phi(\tfrac{1}{2}\,\omega\tau) \tag{13}$$

Then, if μ is pure imaginary, ξ always is bounded, and, for $\mu > 0$, ξ will be bounded (stable) as τ increases if

$$\delta \geq \tfrac{1}{2}\,\mu\omega \tag{14}$$

and ξ will be unbounded (unstable) as τ increases if

$$\delta < \tfrac{1}{2}\,\mu\omega \tag{15}$$

For a given pair of A and B, μ can be found readily from Fig. 2. For a given cylinder, δ remains to be determined. There are many causes of damping, but, according to Ref. 12, the damping force in thin cylinders at free vibration is of the viscous type. Furthermore, experimentally the quantity (δ/π) is found to be equal to 6 cps, which is nearly a constant regardless of the frequency range, internal pressure, and vibration mode. Therefore,

$$\delta/\pi = 6 \text{ cps} \tag{16}$$

may be used for design purposes.

It is not difficult to establish the range of interest for μ once the quantity δ is known. For example, if the highest frequency of the external forces is 120 cps, through substitution of this figure and Eq. (16) into Eq. (14), one gets

$$\mu \leq 2\delta/\omega = \tfrac{6}{120} = 0.05 \tag{17}$$

Therefore, the minimum tolerable value for μ is 0.05.

V. Numerical Results

To determine directly whether or not a system with δ, ν, and r/h is stable when it is subjected to the force system N_0, P_0, a, b, e, f, and ω and assumes all the modes of n and k, proceed as follows:

1) Calculate points A, B according to Eq. (11) with all the possible combinations of those parameters just mentioned.

2) Plot points A, B in the x-y coordinates on the same scale used for Fig. 2. (One such plot is shown in Fig. 3.)

3) Lay the tracing of Fig. 2 over the A-B plot and search for points in the unstable regions. If damping effect is considered, calculate μ according to Eq. (14) and search for unstable points in the unstable regions and above those iso-μ curves.

4) If not a single unstable point is found, the system is stable.

In the following sections, numerical results for several realistic cases are obtained from which the real physical signi-

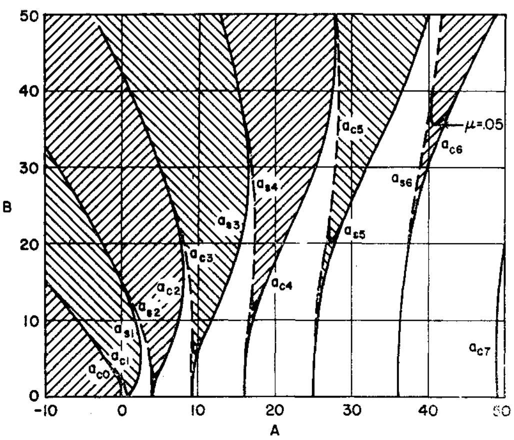

Fig. 1 Stability chart of Mathieu's equation

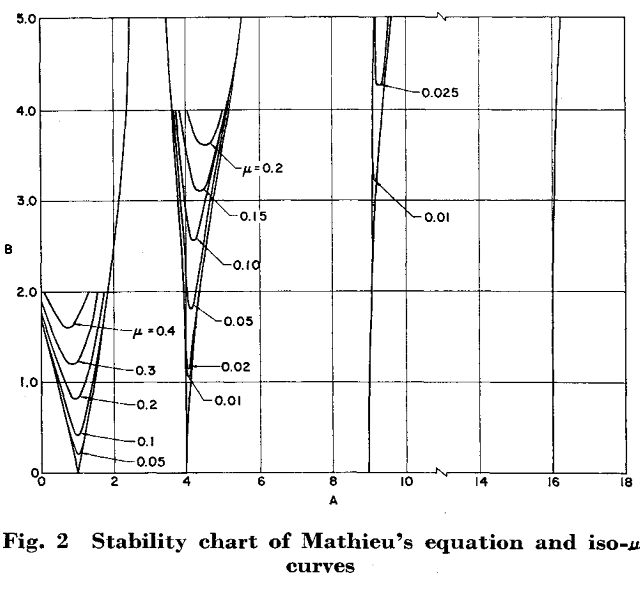

Fig. 2 Stability chart of Mathieu's equation and iso-μ curves

† Note that a real μ will yield an unstable solution for the Mathieu Eq. (10) but will not yield necessarily an unstable solution for the original equation of motion (3) because of the presence of the damping term δ [see also Eq. (5)]

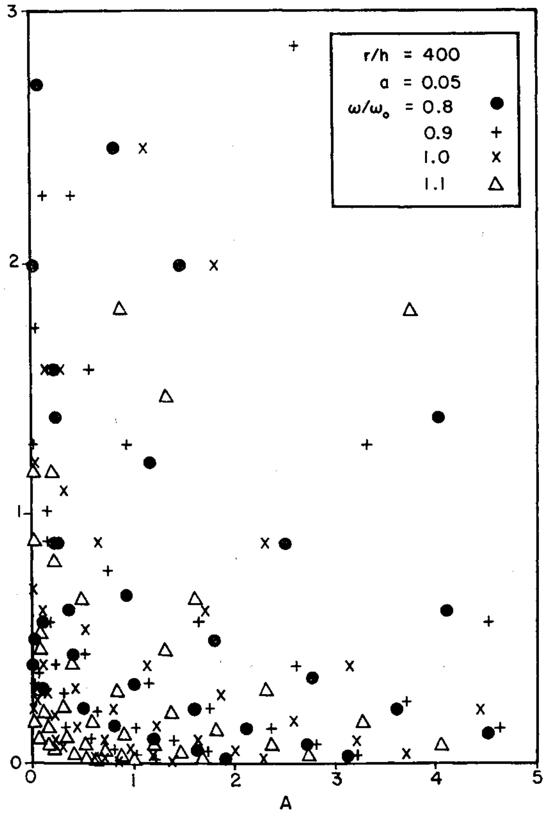

Fig. 3 Values of A and B for cylinders under static and periodic axial loads

ficance of stability problems may be understood better. The cases treated are closed cylinders under static and periodic external pressure, cylinders under static and periodic radial pressure, cylinders under static and periodic axial loads, and an interaction curve for cylinders under varying axial and radial loads. Each problem will be given separate attention; however, certain general details of the calculations may be explained best here. To put the report within reasonable bounds, μ is taken as a constant equal to 0.05 for all cases, although, in practice, there certainly is no reason for such a limitation. With the chosen value of μ, the range of k from 4 to 24 was found to be sufficient for determining unstable points for most of the cases investigated. Observation indicates that a higher value of k brings the A-B point farther away from the unstable regions. (The theory given in Ref. 10, p. 90, also can be used as a guide in selecting the sufficient range of k. This will be explored further in the Appendix.)

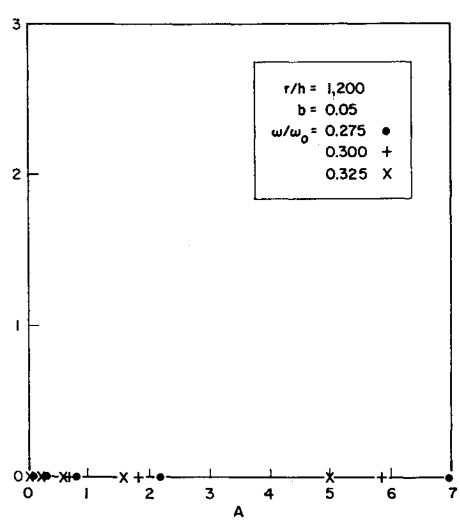

Fig. 4 Values of A and B for cylinders under radial pressure

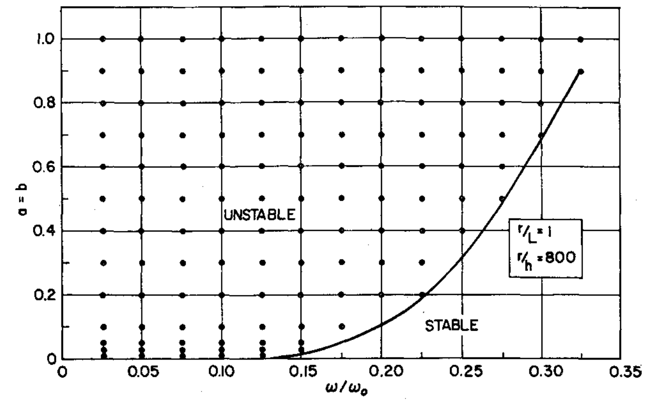

Fig. 5 Stability boundaries for closed cylinders under static and periodic external pressure

The values of A and B given by Eq. (11) first are calculated by IBM 7090; a printout of the x-y plot then is obtained. Samples of such a printout are shown partially for one case in Fig. 3 and fully for another case in Fig. 4. Each point represents a different set of values for n and k. It is apparent that the results in Fig. 3 represent the unstable solution, whereas those in Fig. 4 represent the stable solution.

A. Closed Cylinders under Static and Periodic External Pressure

A cylinder with its ends enclosed but simply supported is subjected to an external pressure having both a static and a periodic component. The static component has a value equal to the static buckling pressure P_s for a finite cylinder, determined in Ref. 13. Hence, by Eq. (7), $N_0 = \frac{1}{2} rP_s$, $P_0 = P_s$, and $e = f = 1$. In the meantime, the amplitude parameters a and b of the periodic forces are given the same variations from zero to 1.0. The nondimensional angular frequency ω/ω_0 ranges from 0.025 to 0.325. Since, for a cylinder under external pressure, the quantity N_0/N_{cr} [see Eq. (11)] is much smaller than P_0/P_{cr} (as $N_{cr} \gg P_{cr}$), and since a cylinder's buckling resistance to the radial pressure is much lower than its resistance to the axial compression, the mode of failure is taken to be the same as that due to the radial pressure, that is, in the axial direction, the mode assumes a half sine-wave.

Two sets of calculations have been carried out. The first set applies to cylinders having a radius-length ratio equal to 1 (for practical design purpose, r/L should be $\sim$1). The various radius-thickness ratios considered and their static buckling values, N_0/N_{cr} and P_0/P_{cr}, are shown in Table 1. (Recall that P_{cr} is the classical buckling pressure of an infinite cylinder.) The stability of the system is determined according to the values of a, b, and ω/ω_0 [Eqs. (7) and (11)]. Whenever an unstable point is found, it is marked on the chart as shown in Fig. 5. Finally, an approximate envelope is drawn which separates the stable and unstable regions.

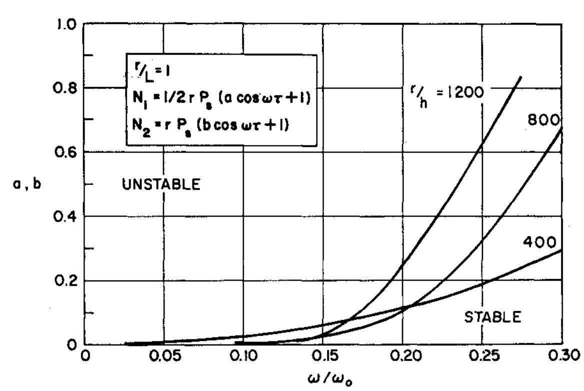

Fig. 6 Stability boundaries for closed cylinders under external pressure (r/L = 1)

Table 1 Static load components

r/h	N_0/N_{cr}	P_0/P_{cr}
400	0.039	69.0
800	0.028	98.6
1200	0.023	120.0

Table 2 Static load components

r/h	N_0/N_{cr}	P_0/P_{cr}
200	0.028	24.6
400	0.021	36.1
800	0.014	49.3
1000	0.012	54.2

Table 3 Static pressure

r/h	P_0/P_{cr}
400	34.5
800	49.2
1200	57.4

Table 4 Static pressure

r/h	P_0/P_{cr}
400	69.1
800	98.7
1200	115.1

Results thus obtained are collected for different values of r/h and given in Fig. 6.

The second set of calculations applies to cylinders having a radius-length ratio equal to $\frac{1}{2}$. The various radius-thickness ratios considered, and their static buckling values, N_0/N_{cr} and P_0/P_{cr}, are shown in Table 2. The final results are given in Fig. 7.

It can be seen from these figures that, in general, a relatively flexible cylinder (with a large r/h) can withhold a higher N_1/P_s or N_2/P_s than a stiffer cylinder (with a smaller r/h). All curves also indicate that at high frequencies the shell can withstand a substantially larger periodic load without loss of stability than it can at low frequencies. The mathematical explanation for this may be found from Eq. (11). When (ω/ω_0) becomes infinite, both A and B approach zero (here δ is assumed to be zero); this is a stable point in Fig. 2. The physical explanation is that the cylinder cannot respond as sensitively when the loading and unloading process becomes too rapid. Of course, if the frequency is too high, the load becomes an impact. This problem is entirely different, and the present results do not apply at all.

B. Cylinders under Static and Periodic Radial Pressure

A cylinder of finite length, simply supported at both ends, is subjected to a static and a periodic radial pressure. The static component of the pressure is assumed to be equal to the static buckling pressure P_s found according to Ref. 13.

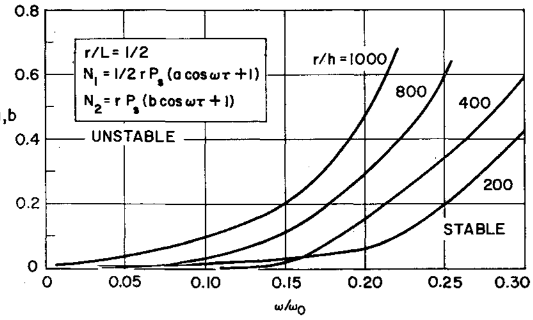

Fig. 7 Stability boundaries for closed cylinders under external pressure ($r/L = \frac{1}{2}$)

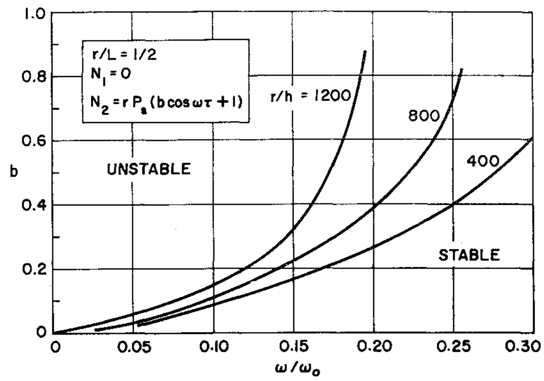

Fig. 8 Stability boundaries for cylinders under radial pressure ($r/L = \frac{1}{2}$)

Hence, by Eq. (7), $f = 1$, $P_0 = P_s$, and $a = e = 0$. The amplitude parameter b of the periodic component of the pressure varies from 0.05 to 1.0, and ω/ω_0 varies from 0.025 to 0.325. The mode of the radial variational displacement is taken as a half sine-wave in the axial direction.

Two groups of calculations are obtained. The first is for $r/L = \frac{1}{2}$. The various radius-thickness ratios considered and their respective values of P_0/P_{cr} are shown in Table 3. The results are shown in Fig. 8. The second group of calculations is obtained for $r/L = 1$. The various radius-thickness ratios and their respective values of P_0/P_{cr} are shown in Table 4. The results are shown in Fig. 9.

As both Figs. 8 and 9 clearly indicate, a cylinder with a higher r/h ratio can withstand a higher N_2/P_s ratio than can one with a lower r/h ratio. Furthermore, cylinders tend to take an appreciably higher periodic pressure without loss of stability at high frequencies than they do at low frequencies.

C. Cylinders under Static and Periodic Axial Loads

Cylinders are assumed to be of medium length so that end conditions have no influence on their behavior. The static component of the load is taken to be equal to the classical static buckling load N_{cr}. The periodic component a in Eq. (7) varies from 0.05, 0.10, . . . , 0.30, and ω/ω_0 varies from 0.025 to 3.0. Also, $k = 4, 6, 8, \ldots, 24$, and $k/n = 0.25, 0.5, \ldots, 1.50$. Numerical results are obtained for $r/h = 400$ and 1200. Results are shown in Fig. 10. It is apparent that the stable regions for this problem are much smaller than those for cylinders under radial pressure. The explanation for this may be found by reviewing Eq. (11). In the expression for B, note that P_0/P_{cr} has a multiplier $(h/r)^2$, whereas N_0/N_{cr} has a multiplier (h/r). Therefore, the axial compression case in general has a larger B than the radial pressure case. Also (from Fig. 2), a point with larger B has more possibility of falling into the region of instability than the one with a smaller B. Thus, the system under

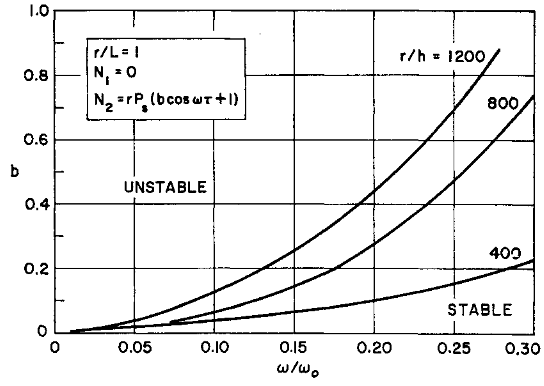

Fig. 9 Stability boundaries for cylinders under radial pressure ($r/L = 1$)

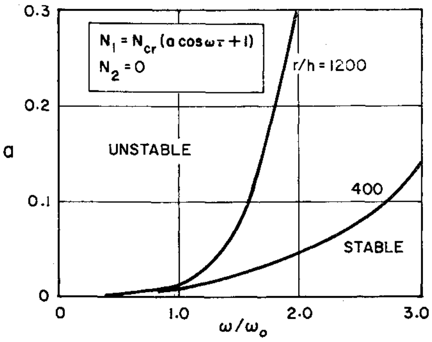

Fig. 10 Stability boundaries for cylinders under static and periodic axial loads

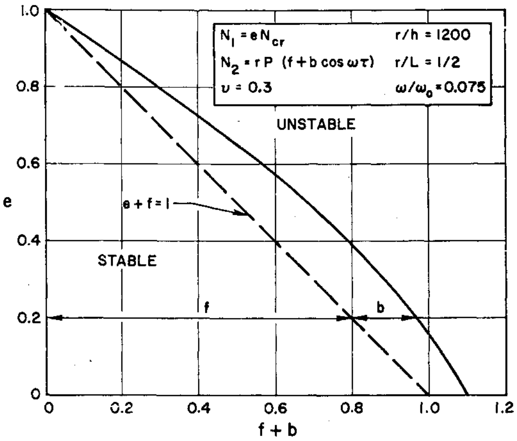

Fig. 11 Interaction curve for cylinders under static axial load and varying radial load

axial load has a smaller stable region. The curves in Fig. 10 also show a common tendency: when ω/ω_0 becomes larger, the region of stability increases.

D. Interaction Curve for Cylinders under Varying Axial Radial Loads

A case problem has been worked out for cylinders with the radius-length ratio $r/L = \frac{1}{2}$, the radius-thickness ratio $r/h = 1200$, and the loading (see Eq. 7) given as

$$N_1 = eN_0 = eN_{cr}$$

$$N_2 = rP_0(f + b\cos\omega\tau) = rP_s(f + b\cos\omega\tau)$$

Here, $P_s = 57.4\,P_{cr}$, as found in the previous case. According to Ref. 14, the relationship between N_1 and rP_sf follows almost a straight line variation in the static case. For the present case, the given load is assumed such that e and f are related by the straight dotted line in Fig. 11. The other factors that enter into the calculation are $\nu = 0.3$; $\omega/\omega_0 = 0.075$; $k = 4,6,8,\ldots,24$; and $n = \pi/2, \pi, \frac{3}{2}\pi, \ldots, 6\frac{1}{2}\pi$. The final result is shown by the solid curve in Fig. 11. The loads found in the region above the curve will be unstable for the system, whereas those below the curve will be stable.

VI. Conclusion

Based on the method of solution outlined at the beginning, the regions of stability are established explicitly for cylinders under various combinations of loads and according to the geometry of the cylinder and the frequency and intensity of the load. To put the work within manageable bounds, a value of $\mu = 0.05$ was taken for all cases [cf., Eq. (17)]. The explicit solutions obtained here, so far, indicate the following characteristics of the problem which appear distinct and in common:

1) A cylinder designed according to the static buckling load can withstand an additional periodic load if the magnitude of this load and its frequency are within the found region of stability.

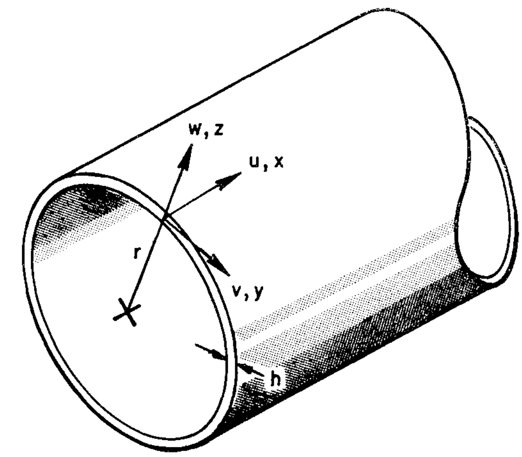

Fig. 12 Coordinate system and displacements

2) The ratio of additional periodic load to the static buckling load was found to be greater for a thin cylinder than for a thicker cylinder of the same radius.

3) The resistance of a cylinder to loads in excess of the static buckling load increases with increasing frequency of loading.

Appendix: Solution of Eq. (10), $A \gg (-B) > 0$

Recalling from Eq. (4) that, if k is large, g is of the order of k^4, and from Eq. (11) that A will be of the order of $(h/r)^2k^4$ and B of $-k^2(h/r)^2$, it can be seen that $A \gg (-B)$. [The same will be true if n is large enough so that $n^2 \gg (r/h)$.] According to Ref. 10, p. 91, if the condition $A \gg (-B)$ is met, the solution for Eq. (10) may be approximated as

$$\eta = Q \sum_{m=-m_0}^{m_0} J_m\left(\frac{B}{4}A^{-1/2}\right)\cos\left[(A^{1/2} - 2m)t - \alpha\right]$$

where Q and α are arbitrary constants, and $A^{1/2} \gg m_0$. Since the cosine term is bounded for all values of t, the A and B associated with η must lie in a stable region of the plane (see Fig. 1).

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